LEGENDRIAN VERTICAL CIRCLES IN SMALL SEIFERT SPACES

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Abstract. We discuss the relations between the e_0 invariants of a small Seifert space and the twisting numbers of Legendrian vertical circles in it.

1. Introduction

A contact structure ξ on an oriented 3-manifold M is a nowhere integrable tangent plane distribution, i.e., near any point of M, ξ is defined locally by a 1-form α, s.t., $\alpha \wedge d\alpha \neq 0$. Note that the orientation of M given by $\alpha \wedge d\alpha$ depends only on ξ, not on the choice of α. ξ is said to be positive if this orientation agrees with the native orientation of M, and negative if not. A contact structure ξ is said to be co-orientable if ξ is defined globally by a 1-form α. Clearly, an co-orientable contact structure is orientable as a plane distribution, and a choice of α determines an orientation of ξ. Unless otherwise specified, all contact structures in this paper will be co-oriented and positive, i.e., with a prescribed defining form α such that $\alpha \wedge d\alpha > 0$. A curve in M is said to be Legendrian if it is tangent to ξ everywhere. ξ is said to be overtwisted if there is an embedded disk D in M such that ∂D is Legendrian, but D is transversal to ξ along ∂D. A contact structure that is not overtwisted is called tight. Overtwisted contact structures appear to be very "soft". It is proven by Eliashberg in [2] that two overtwisted contact structures are isotopic *iff* they are homotopic as tangent plane distributions. Tight contact structures are more rigid. Classifications of tight contact structures up to isotopy are only known for very limited classes of 3-manifolds. (See, e.g., [3], [4], [5], [6], [7], [12], [13], [15].)

A small Seifert space is a Seifert fibred space with 3 singular fibers over S^2. Any regular fiber f in a small Seifert space $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ admits a canonical framing given by pulling back an arc in the base S^2 containing the projection of f. An embedded circle in $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ is said to be vertical if it is isotopic to a regular fiber. Any vertical circle inherits a canonical framing from the canonical framing of regular fibers. We call this framing Fr.

Definition 1.1. Let ξ be a contact structure on a small Seifert space $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, and L a Legendrian vertical circle in (M, ξ). The twisting number $t(L)$ of L is the twisting number of $\xi|_L$ along L relative to the canonical framing Fr of L.

In [1], Colin, Giroux and Honda divided the tight contact structures on a small Seifert space into two types: those for which there exists a Legendrian vertical circle with twisting number 0, and those for which no Legendrian vertical circles with twisting number 0 exist. It is proven in [1] that, up to isotopy, the number of tight contact

structures of the first type is always finite, and, unless the small Seifert space is also a torus bundle, the number of tight contact structures of the second type is also finite. Their work gives in principle a method to estimate roughly the upper bound of the number of tight contact structures on a small Seifert space. In the present paper, we demonstrate that most small Seifert spaces admit only one of the two types of tight contact structures. To make our claim precise, we need the following invariant. (See, e.g., [10].)

Definition 1.2. For a small Seifert space $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, define $e_0(M) = \lfloor \frac{q_1}{p_1} \rfloor + \lfloor \frac{q_2}{p_2} \rfloor + \lfloor \frac{q_3}{p_3} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer not greater than x.

Clearly, $e_0(M)$ is an invariant of M, i.e., it does not depend on the choice of the representatives $(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$. Now we can formulate our claim precisely in the following theorems.

Theorem 1.3. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space. If $e_0(M) \geq 0$, then every tight contact structure on M admits a Legendrian vertical circle with twisting number 0.*

Theorem 1.4. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space. If $e_0(M) \leq -2$, then no tight contact structures on M admit Legendrian vertical circles with twisting number 0.*

In particular, Theorem 1.4 means that, for any small Seifert space $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, either M or $-M$ does not admit tight contact structures for which there exists a Legendrian vertical circle with twisting number 0, where $-M$ is M with reversed orientation. This is because that $e_0(M) + e_0(-M) = -3$, and, hence, one of $e_0(M)$ and $e_0(-M)$ has to be less than or equal to -2.

It turns out that the case when $e_0(M) = -1$ is the most difficult. Only very weak partial results are known. For example, in [7], Ghiggini and Schönenberger proved that, when $r \leq \frac{1}{5}$, no tight contact structures on the small Seifert space $M(r, \frac{1}{3}, -\frac{1}{2})$ admit Legendrian vertical circles with twisting number 0.

We have the following results about the case when $e_0(M) = -1$.

Theorem 1.5. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space such that $0 < q_1 < p_1$, $0 < q_2 < p_2$ and $-p_3 < q_3 < 0$.*

(1) If $\frac{q_1}{p_1} + \frac{q_3}{p_3} \geq 0$ or $\frac{q_2}{p_2} + \frac{q_3}{p_3} \geq 0$ or $\frac{q_1}{p_1} + \frac{q_2}{p_2} \geq 1$, then every tight contact structure on M admits a Legendrian vertical circle with twisting number 0.

(2) If $q_3 = -1$, $\frac{q_1}{p_1} < \frac{1}{2p_3 - 1}$ and $\frac{q_2}{p_2} < \frac{1}{2p_3}$, then no tight contact structures on M admit Legendrian vertical circles with twisting number 0.

(3) If $q_1 = q_2 = 1$ and $p_1, p_2 > -2\lfloor \frac{p_3}{q_3} \rfloor$, then no tight contact structures on M admit Legendrian vertical circles with twisting number 0.

To understand the proofs in this paper, readers need to be familiar with the techniques developed by Giroux in [8] and Honda in [12]. For those who are not, there is a concise introduction to these techniques in [7].

2. The $e_0 \geq 0$ Case

The $e_0 \geq 0$ case is the simplest case. Theorem 1.3 is a special case of Lemma 2.2 below, which also implies part (1) of Theorem 1.5.

In the rest of this paper, Σ means a three hole sphere; Σ_0 means a properly embedded three hole sphere in $\Sigma \times S^1$ isotopic to $\Sigma \times \{\text{pt}\}$. Let $-\partial\Sigma \times S^1 = T_1 + T_2 + T_3$, where the "$-$" sign means reversing the orientation. We identify T_i to $\mathbf{R^2}/\mathbf{Z^2}$ by identifying the corresponding component of $-\partial\Sigma \times \{\text{pt}\}$ to $(1,0)^T$, and $\{\text{pt}\} \times S^1$ to $(0,1)^T$. An embedded circle in T_i is called vertical if it is essential and has slope ∞.

The following lemma is purely technical.

Lemma 2.1. *Let ξ be a tight contact structure on $\Sigma \times S^1$. Assume that each T_i is convex with two dividing curves of slope s_i. Then there exist collar neighborhoods $T_1 \times I$ and $T_2 \times I$ of T_1 and T_2 and a properly embedded vertical convex annulus A in $(\Sigma \times S^1) \setminus (T_1 \times I \cup T_2 \times I)$ connecting $T_1 \times \{1\}$ to $T_2 \times \{1\}$ with Legendrian boundary satisfying that following:*

1. *$T_1 \times I$ and $T_2 \times I$ are mutually disjoint and disjoint from T_3;*
2. *for $i = 1, 2$, $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is convex with two dividing curves of slope $s'_i \leq s_i$;*
3. *A has no ∂-parallel dividing curves, and the Legendrian boundary of A intersects the dividing sets of $T_1 \times \{1\}$ and $T_2 \times \{1\}$ efficiently.*

Proof. If both s_1 and s_2 are ∞, then we can isotope T_1 and T_2 slightly to make them to have vertical Legendrian divides. Connect a Legendrian divide of T_1 to a Legendrian divide of T_2 by a properly embedded vertical convex annlus A. Then we are done. If $s_1 = \infty$, but s_2 is finite, then we make T_1 to have vertical Legendrian divides, and T_2 to have vertical Legendrian rulings. Connect a Legendrian divide of T_1 to a Legendrian ruling of T_2 by a properly embedded vertical convex annlus B. Then no dividing curves of B intersects $B \cap T_1$. And we can decrease s_2 to ∞ by isotoping T_2 across the dividing curves of B starting and ending on $B \cap T_2$ through bypass adding. We can keep T_2 disjoint from both T_1 and T_3 through out this isotopy since bypass adding can be done in a small neighborhood of the bypass and the original surface. The we are back to the case when s_1 and s_2 are both ∞.

Assume $s_i = \frac{q_i}{p_i}$ is finite for $i = 1, 2$, where $p_i > 0$. First, we isotope T_1 and T_2 slightly so that they have vertical Legendrian rulings. Note that the Legendrian rulings always intersect dividing curves efficiently. Then connect a Legendrian ruling of T_1 to a Legendrian ruling of T_2 by a properly embedded vertical convex annulus A in $\Sigma \times S^1$. If A has no ∂-parallel dividing curves, then we are done. If A has a ∂-parallel dividing curve, say on the T_1 side, then, after possibly isotoping A slightly, we can assume there is a bypass of T_1 on A. Adding this bypass to T_1, we get a convex torus T'_1 with two dividing curves in $\Sigma \times S^1$ that co-bounds a collar neighborhood of T_1. We can make T'_1 to have vertical Legendrian ruling. By Lemma 3.5 of [12], we have that T'_1 has two dividing curves of the slope $s'_1 = \frac{q'_1}{p'_1} < s_1$, where $0 \leq p'_1 < p_1$. Now we delete the thickened torus between T_1 and T'_1 from $\Sigma \times S^1$, and repeat the procedure above. This whole process will stop in less than $p_1 + p_2$ steps, i.e, we can either find the collar neighborhoods and the annulus with the required properties, or

force one of s_1 and s_2 to decrease to ∞. But the lemma is proved in the latter case. This finishes the proof. $\square$

Lemma 2.2. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space such that $\frac{q_1}{p_1}, \frac{q_2}{p_2} > 0$ and $\frac{q_1}{p_1} + \frac{q_3}{p_3} \geq 0$. Then every tight contact structure on M admits a Legendrian vertical circle with twisting number 0.*

Proof. For $i = 1, 2, 3$, let $V_i = D^2 \times S^1$, and identify ∂V_i with $\mathbf{R^2}/\mathbf{Z^2}$ by identifying a meridian $\partial D^2 \times \{\text{pt}\}$ with $(1, 0)^T$ and a longitude $\{\text{pt}\} \times S^1$ with $(0, 1)^T$. Choose $u_i, v_i \in \mathbf{Z}$ such that $0 < u_i < p_i$ and $p_i v_i + q_i u_i = 1$ for $i = 1, 2, 3$. Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \to T_i$ by

$$\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & v_i \end{pmatrix},$$

where T_i and the coordinates on it are defined above. Then

$$M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).$$

Let ξ be a tight contact structure on M. We first isotope ξ to make each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the $\frac{q_i}{p_i}$-singular fiber with twisting number $n_i < 0$, i.e., ∂V_i is convex with two dividing curves each of which has slope $\frac{1}{n_i}$ when measured in the coordinates of ∂V_i given above. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \frac{-n_i q_i + v_i}{n_i p_i + u_i} = -\frac{q_i}{p_i} + \frac{1}{p_i(n_i p_i + u_i)} < -\frac{q_i}{p_i}.$$

By Lemma 2.1, we can thicken V_1 and V_2 to V_1' and V_2' such that

(1) V_1', V_2' and V_3 are pairwise disjoint;
(2) for $i = 1, 2$, $T_i' = \varphi_i(\partial V_i')$ is convex with two dividing curves of slope $s_i' = -\frac{q_i'}{p} \leq s_i$, where $p, q_i' > 0$;
(3) there exists a properly embedded vertical convex annulus A connecting T_1' to T_2' that has no ∂-parallel dividing curves, and the Legendrian boundary of A intersects the dividing sets of these tori efficiently.

If none of the dividing curves of A is an arc connecting the two components of ∂A, then, by the Legendrian Realization Principle ([8], [12]), we can isotope A to make a vertical circle L on A which is disjoint from the dividing curves Legendrian. Note that A gives the canonical framing of L, and the twisting number of $\xi|_L$ relative to $TA|_L$ is 0 by Proposition 4.5 of [16]. So $t(L) = 0$.

If there are dividing curves connecting the two components of ∂A, then cut $M \backslash (V_1' \cup V_2' \cup V_3)$ open along A. We get an embedded thickened torus $T_3 \times I$ in M such that $T_3 \times \{0\} = T_3$, and $T_3 \times \{1\}$ is convex with two dividing curves of slope $s_3' = \frac{q_1' + q_2' - 1}{p}$. Note that

$$s_3' = \frac{q_1' + q_2' - 1}{p} \geq \frac{q_1'}{p} \geq -s_1 > \frac{q_1}{p_1} \geq -\frac{q_3}{p_3} > s_3.$$

According to Theorem 4.16 of [12], there exists a convex torus T in $T_3 \times I$ parallel to T_3 with vertical dividing curves. We can then isotope T to make it in standard form.

Then a Legendrian divide of T is a Legendrian vertical circle with twisting number 0. $\square$

Proof of Theorem 1.3 and Theorem 1.5(1). If $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ satisfies that $e_0(M) \geq 0$, then we can assume that $\frac{q_i}{p_i} > 0$ for $i = 1, 2, 3$. It's then clear that $\frac{q_1}{p_1}, \frac{q_2}{p_2} > 0$ and $\frac{q_1}{p_1} + \frac{q_3}{p_3} > 0$. Thus, Lemma 2.2 implies Theorem 1.3.

Now we assume $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ is a small Seifert space such that $0 < q_1 < p_1$, $0 < q_2 < p_2$ and $-p_3 < q_3 < 0$. If $\frac{q_1}{p_1} + \frac{q_3}{p_3} \geq 0$ or $\frac{q_2}{p_2} + \frac{q_3}{p_3} \geq 0$, then Lemma 2.2 applies directly. If $\frac{q_1}{p_1} + \frac{q_2}{p_2} \geq 1$, we apply Lemma 2.2 to $M = M(\frac{q_1}{p_1}, \frac{q_3}{p_3} + 1, \frac{q_2}{p_2} - 1)$. This proves Theorem 1.5(1). $\square$

3. The $e_0 \leq -2$ Case

Definition 3.1. Let ξ be a contact structure on $\Sigma \times S^1$. ξ is said to be inappropriate if ξ is overtwisted, or there exists an embedded $T^2 \times I$ with convex boundary and I-twisting at least π such that $T^2 \times \{0\}$ is isotopic to one of the T_i's. ξ is called appropriate if it is not inappropriate.

Lemma 3.2. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space, and ξ a tight contact structures on M. Suppose that V_1, V_2, V_3 are tubular neighborhoods of the three singular fibers, and $\Sigma \times S^1 = M \setminus (V_1 \cup V_2 \cup V_3)$. Then $\xi|_{\Sigma \times S^1}$ is appropriate.*

Proof. Without loss of generality, we assume ∂V_i is identified with T_i by the diffeomorphism φ_i. $\xi|_{\Sigma \times S^1}$ is clearly tight. If it is inappropriate, then there exists an embedded $T^2 \times I$ with convex boundary and I-twisting at least π such that $T^2 \times \{0\}$ is isotopic to one of the T_i's. Let's say $T^2 \times \{0\}$ is isotopic to T_1. $T^2 \times I$ has I-twisting at least π implies that, for any rational slope s, there is a convex torus T_0 contained in $T^2 \times I$ isotopic to T_1 that has dividing curves of slope s. Specially, we let m be a meridian of ∂V_1, and s the slope of $\varphi_1(m)$. Then the above fact means that we can thicken V_1 so that ∂V_1 has dividing curves isotopic to its meridians, which implies that the thickened V_1 is overtwisted. This contradicts the tightness of ξ. Thus, $\xi|_{\Sigma \times S^1}$ must be appropriate. $\square$

Lemma 3.3 ([5], Lemma 10)**.** *Let ξ be an appropriate contact structure on $\Sigma \times S^1$ such that all three boundary components of $\Sigma \times S^1$ are convex, and the dividing set of each of these boundary components consists of two vertical circles. If Σ_0 is convex with Legendrian boundary that intersects the dividing set of $\partial\Sigma \times S^1$ efficiently, then the dividing set of Σ_0 consists of three properly embedded arcs, each of which connects a different pair of components of $\partial\Sigma_0$.*

The following lemma is a special case of Proposition 6.4 of [1], which also appears in [9] and [13].

Lemma 3.4 ([1], [9], [13])**.** *Isotopy classes of tight contact structures on $\Sigma \times S^1$ such that all three boundary components of $\Sigma \times S^1$ are convex, and the dividing set of each of these boundary components consists of two vertical circles are in 1-1 correspondence*

with isotopy classes of embedded multi-curves on Σ with 2 fixed end points on each component of $\partial\Sigma$ that have no homotopically trivial components.

The correspondence here is induced by the following mapping: Given a tight contact structure ξ on $\Sigma \times S^1$, isotope Σ_0 to make it convex with Legendrian boundary intersecting the dividing set of $\partial\Sigma \times S^1$ efficiently. Then map ξ to the isotopy class of the dividing curves of $\Sigma_0(\cong \Sigma)$.

The following lemma from [7] plays a key role in the proof of Theorem 1. For the convenience of readers, we give a detailed proof here.

Lemma 3.5 ([7], Lemma 36)**.** *Let ξ be an appropriate contact structure on $\Sigma \times S^1$. Suppose that $-\partial\Sigma \times S^1 = T_1 + T_2 + T_3$ is convex and such that each of T_1 and T_2 has vertical Legendrian rulings and two dividing curves of slope $-\frac{1}{n}$, where $n \in \mathbf{Z}^{>0}$, and T_3 has two vertical dividing curves. Let $T_1 \times I$ and $T_2 \times I$ be collar neighborhoods of T_1 and T_2 that are mutually disjoint and disjoint from T_3, and such that, for $i = 1, 2$, $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is convex with dividing set consisting of two vertical circles. If $\xi|_{T_1\times I}$ and $\xi|_{T_2\times I}$ are both isotopic to a given minimal twisting tight contact structure η on $T^2 \times I$ relative to the boundary, then there exists a properly embedded convex vertical annulus A with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap T_1) \cup (A \cap T_2)$ intersects the dividing sets of T_1 and T_2 efficiently.*

Proof. Let $\Sigma' \times S^1 = (\Sigma \times S^1) \setminus [(T_1 \times [0, 1)) \cup (T_2 \times [0, 1))]$, and Σ'_0 a properly embedded convex surface in $\Sigma' \times S^1$ isotopic to $\Sigma' \times \{\mathrm{pt}\}$ that has Legendrian boundary intersecting the dividing set of $\partial\Sigma' \times S^1$ efficiently. Since $\xi|_{\Sigma'\times S^1}$ is appropriate, the dividing set of Σ'_0 consists of three properly embedded arc, each of which connects a different pair of boundary components of Σ'_0. Up to isotopy relative to $\partial\Sigma'_0$, there are infinitely many such multi-arcs on Σ'_0. But, up to isotopy of Σ'_0 which leaves $\partial\Sigma'_0$ invariant, there are only two, each represented by a diagram in Figure 1 below. Such an isotopy of Σ'_0 extend to an isotopy of $\Sigma' \times S^1$ which, when restricted to a component of $\partial\Sigma' \times S^1$, is a horizontal rotation. Thus, up to isotopy of $\Sigma' \times S^1$, which, when restricted to any of the components of $\partial\Sigma' \times S^1$, is a horizontal rotation, there are only two appropriate contact structures on $\Sigma' \times S^1$. Now let Φ_t be such an isotopy of $\Sigma' \times S^1$ changing $\xi|_{\Sigma'\times S^1}$ to one of the two standard appropriate contact structures. We extend Φ_t to an isotopy $\widetilde{\Phi}_t$ of $\Sigma \times S^1$, which fixes a neighborhood of $T_1 \cup T_2$, and leaves $T_1 \times I$, $T_2 \times I$ and $\Sigma' \times S^1$ invariant. Note that the relative Euler class of $\xi|_{T_i\times I}$ is $(2k - n, 0)^T$, where k is the number of positive basic slices contained in $(T^2 \times I, \eta)$, and is invariant under $\widetilde{\Phi}_t|_{T_i\times I}$. So $\xi|_{T_i\times I}$ and $\widetilde{\Phi}_{1*}(\xi)|_{T_i\times I}$ have the same relative Euler class, and are both continued fraction blocks satisfying the same boundary condition. According to the classification of tight contact structures on $T^2 \times I$, $\xi|_{T_i\times I}$ and $\widetilde{\Phi}_{1*}(\xi)|_{T_i\times I}$ are isotopic relative to boundary. So $\widetilde{\Phi}_{1*}(\xi)$ satisfies the conditions given in the lemma, and is of one of the two standard form. Thus, up to isotopy fixing T_1, T_2 and leaving T_3 invariant, there are only two appropriate contact structures on $\Sigma \times S^1$ satisfying the given conditions. Rotating the diagram on the left of Figure 1 by $180°$ induces a self-diffeomorphism of $\Sigma \times S^1$ mapping T_1 to T_2 and changing the dividing set of Σ'_0 on the left of Figure 1 to the one on the right. So this self-diffeomorphism is isotopic to a contactomorphism between the two standard appropriate contact structures on

$\Sigma \times S^1$. Hence, up to contactomorphism, there is only one such appropriate contact structure on $\Sigma \times S^1$. Thus, we can show the existence of an annulus with the required properties by explicitly constructing such an annulus in a model contact structure on $\Sigma \times S^1$ which satisfies the given conditions.

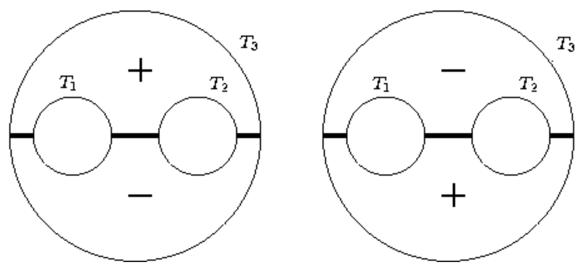

FIGURE 1.

Consider the minimal twisting tight contact structure η on the thickened torus $T^2 \times I$. Note that the vertical Legendrian rulings of $T^2 \times \{0\}$ intersect its dividing curves efficiently. Without loss of generality, we assume that $T^2 \times \{1\}$ has horizontal Legendrian rulings and two vertical Legendrian dividings. We further assume that, for a small $\varepsilon > 0$, $\eta|_{T^2\times[0,\varepsilon]}$ is invariant in the I direction. This is legitimate since $T^2 \times \{0\}$ is convex. So $T^2 \times \{\frac{\varepsilon}{2}\}$ is also a convex torus with vertical Legendrian rulings and dividing curves of slope $-\frac{1}{n}$. Let L be a Legendrian ruling of $T^2 \times \{\frac{\varepsilon}{2}\}$. Since the twisting number of $\eta|_L$ relative to the framing given by $T^2 \times \{\frac{\varepsilon}{2}\}$ is $-n$, we can find a standard neighborhood U of L in $T^2 \times (0, \varepsilon)$ such that ∂U is convex with vertical Legendrian ruling and two dividing curves of slope $-\frac{1}{n}$. Now, we set $\Sigma \times S^1 = (T^2 \times I) \setminus U$, where $T_1 = T^2 \times \{0\}$, $T_2 = \partial U$ and $T_3 = -T^2 \times \{1\}$, and let $\xi = \eta|_{\Sigma\times S^1}$. Since η is tight, so is ξ. And there are no embedded thickened tori in $\Sigma \times S^1$ with convex boundary isotopic to T_2 and I-twisting at least π. Otherwise, L would have an overtwisted neighborhood in $T^2 \times I$, which contradicts the tightness of η. Also, since the I-twisting of η is less than π, there exists no embedded thickened tori in $\Sigma \times S^1$ with convex boundary isotopic to T_1 or T_3 and I-twisting at least π. Thus, ξ is appropriate. Now we choose a vertical convex annulus A_1 in $\Sigma \times S^1$ connecting a Legendrian ruling of T_1 to a Legendrian divide of T_3, and a vertical convex annulus A_2 in $\Sigma \times S^1$ connecting a Legendrian ruling of T_2 to the other Legendrian divide of T_3 such that $(T_1 \cup A_1) \cap (T_2 \cup A_2) = \phi$. The dividing set of A_i consists of n arcs starting and ending on $A_i \cap T_i$. We can find a collar neighborhood $T_i \times I$ of T_i, for which $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is convex with dividing set consisting of two circles of slope ∞, by isotoping T_i to engulf all the dividing curves of A_i through bypass adding. Since bypass adding can be done in a small neighborhood of the original surface and the bypass, we can make $T_1 \times I$ and $T_2 \times I$ mutually disjoint and disjoint from T_3. Note that both $T_1 \times I$ and $T_2 \times I$ are minimal twisting. So they are continued fraction blocks satisfying the boundary conditions specified above. Let k_i be the number of positive slices in $T_i \times I$, and $B_i = A_i \cap (T_i \times I)$. Then $2k_i - n = \chi((B_i)_+) - \chi((B_i)_-) = \chi((A_i)_+) - \chi((A_i)_-)$. But $\chi((A_1)_+) - \chi((A_1)_-) = 2k - n$, where k is the number of positive basic slices

contained in $(T^2\times I,\eta)$. So $k_1=k$. And, since $\eta|_{T^2\times(0,\varepsilon)}$ is I-invariant, we can extend A_2 to a vertical annulus $\widetilde{A}_2$ in $T^2\times I$ starting at a Legendrian ruling of T_1 and such that $\chi((\widetilde{A}_2)_+)-\chi((\widetilde{A}_2)_-)=\chi((A_2)_+)-\chi((A_2)_-)$. Clearly, $2k-n=\chi((\widetilde{A}_2)_+)-\chi((\widetilde{A}_2)_-)$. So $k_2=k$. Thus, $k_1=k_2=k$. But the isotopy type of a continued fraction block is determined by the number of positive slices in it. Thus, $\xi|_{T_1\times I}$, $\xi|_{T_2\times I}$ and η are isotopic relative to boundary. So our $(\Sigma\times S^1,\xi)$ is indeed a legitimate model. Now we connect a Legendrian ruling of T_1 and a Legendrian ruling of T_2 by a vertical convex annulus A which is contained in $(T^2\times[0,\varepsilon])\setminus U$. Then ∂A intersects the dividing sets of T_1 and T_2 efficiently. If A has ∂-parallel diving curves, then $(T^2\times[0,\varepsilon])$ has non-zero I-twisting, which contradicts our choice of the slice $(T^2\times[0,\varepsilon])$. Thus, A has no ∂-parallel diving curves. $\square$

Now we are in the position to prove Theorem 1.4.

Proof of Theorem 1.4. Let $M=M(\frac{q_1}{p_1},\frac{q_2}{p_2},\frac{q_3}{p_3})$ be a small Seifert space with $e_0(M)\leq -2$. Without loss of generality, we assume that $p_1,p_2,p_3>1$, $0<q_1<p_1$, and $q_2,q_3<0$. For $i=1,2,3$, let $V_i=D^2\times S^1$, and identify ∂V_i with $\mathbf{R^2}/\mathbf{Z^2}$ by identifying a meridian $\partial D^2\times\{\text{pt}\}$ with $(1,0)^T$ and a longitude $\{\text{pt}\}\times S^1$ with $(0,1)^T$. Choose $u_i,v_i\in\mathbf{Z}$ such that $p_iv_i+q_iu_i=1$ for $i=1,2,3$. Define an orientation preserving diffeomorphism $\varphi_i:\partial V_i\to T_i$ by

$$\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & v_i \end{pmatrix},$$

where T_i and the coordinates on it are defined above. Then

$$M=M(\frac{q_1}{p_1},\frac{q_2}{p_2},\frac{q_3}{p_3})\cong(\Sigma\times S^1)\cup_{(\varphi_1\cup\varphi_2\cup\varphi_3)}(V_1\cup V_2\cup V_3).$$

Assume that ξ is a tight contact structure on M for which there exits a Legendrian vertical circle L in M with twisting number $t(L)=0$. We first isotope ξ to make $L=\{\text{pt}\}\times S^1\subset\Sigma\times S^1$, and each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the $\frac{q_i}{p_i}$-singular fiber with twisting number $n_i<0$, i.e., ∂V_i is convex with two dividing curves each of which has slope $\frac{1}{n_i}$ when measured in the coordinates of ∂V_i given above. Let s_i be the slope of the dividing curves of $T_i=\partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \frac{-n_iq_i+v_i}{n_ip_i+u_i} = -\frac{q_i}{p_i}+\frac{1}{p_i(n_ip_i+u_i)}.$$

From our choice of p_i and q_i, one can see that $-1\leq s_1\leq 0$ and $0\leq s_2,s_3<\infty$. Now, without affecting the properties of L and V_i asserted above, we can further isotope the contact structure ξ to make the Legendrian rulings of T_i to have slope ∞ when measured in the coordinates of T_i.

Pick a Legendrian ruling $\widetilde{L}_i$ on each T_i, and connect L to $\widetilde{L}_i$ by a vertical convex annulus A_i such that $A_i\cap A_j=L$ when $i\neq j$. Let Γ_{A_i} be the dividing set of A_i. Since A_i gives the canonical framing Fr of L, we know that the twisting number of $\xi|_L$ relative to $TA_i|_L$ is 0. This means that $\Gamma_{A_i}\cap L=\phi$. But $\Gamma_{A_i}\cap\widetilde{L}_i\neq\phi$. There are dividing curves of A_i starting and ending on $\widetilde{L}_i$. According to Lemma 3.15 of [12], we

can find an embedded minimal twisting slice $T_i \times I$ in $\Sigma \times S^1$, for which $T_i \times \{0\} = T_i$, $T_i \times \{1\}$ is convex with two vertical dividing curves, by isotoping T_i to engulf all the dividing curves of A_i starting and ending on $\widetilde{L}_i$ through bypass adding. Since bypass adding can be done in a small neighborhood of the bypass and the original surface, and the bypasses from different A_i's are mutually disjoint, we can make $T_i \times I$'s pairwise disjoint. By Corollary 4.16 of [12], we can find a convex torus in $T_i \times (0, 1)$ isotopic to T_i that has two dividing curves of the slope -1. Without loss of generality, we assume that this torus is $T_i \times \{\frac{1}{2}\}$. Moreover, for $i = 2, 3$, we can find another convex torus, say $T_i \times \{\frac{1}{4}\}$, in $T_i \times (0, \frac{1}{2})$ isotopic to T_i with two dividing curves of slope 0.

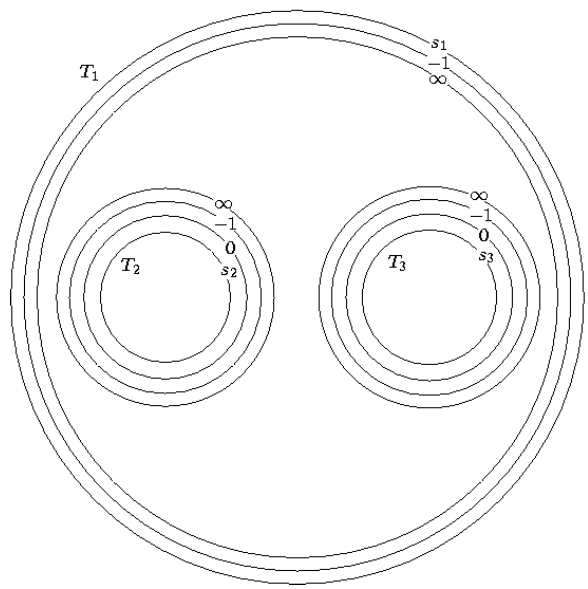

FIGURE 2.

Since the slice $T_i \times I$ is minimal twisting, so is any of its sub-slices. Let's consider the thickened tori $T_i \times [\frac{1}{2}, 1]$. All of these have the same boundary condition, and are minimal twisting. There are only two such tight contact structures up to isotopy relative to boundary. So two of these have to be isotopic relative to boundary. There are 3 cases.

Case 1. $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ are isotopic. We apply Lemma 3.5 to

$$\Sigma' \times S^1 = (\Sigma \times S^1) \setminus (T_1 \times [0, \frac{1}{2}) \cup T_2 \times [0, \frac{1}{2}) \cup T_3 \times [0, 1)).$$

Then there exists a vertical convex annulus A connecting $T_1 \times \{\frac{1}{2}\}$ and $T_2 \times \{\frac{1}{2}\}$ with no ∂-parallel dividing curves that has Legendrian boundary intersecting the dividing sets of these tori efficiently. We can extend A across $T_2 \times [\frac{1}{4}, \frac{1}{2}]$ to a convex annulus $\widetilde{A}$ connecting $T_1 \times \{\frac{1}{2}\}$ and $T_2 \times \{\frac{1}{4}\}$ with Legendrian boundary intersecting the dividing sets of these two tori efficiently. Since $T_2 \times [\frac{1}{4}, \frac{1}{2}]$ is minimal twisting, $\widetilde{A} \setminus A$ has no ∂-parallel dividing curves. Thus, $\widetilde{A}$ has no ∂-parallel dividing curves either. Cut

$(\Sigma\times S^1)\setminus(T_1\times[0,\frac{1}{2})\cup T_2\times[0,\frac{1}{4})\cup T_3\times[0,1))$ along $\widetilde{A}$, and round the edges. We get a thickened torus $T_3\times[1,2]$ embedded in $\Sigma\times S^1$ with convex boundary. The dividing set of $T_3\times\{2\}$ consists two circles of slope 0. Now we can see that the thickened torus $T_3\times[0,2]$ has I-twisting at least π since the dividing curves of $T_3\times\{\frac{1}{4}\}$ and $T_3\times\{2\}$ have slope 0 and those of $T_3\times\{1\}$ have slope ∞. Thus, $\Sigma\times S^1$ is inappropriate. This is a contradiction.

Case 2. $T_1\times[\frac{1}{2},1]$ and $T_3\times[\frac{1}{2},1]$ are isotopic. The proof for this case is identical to that of Case 1 except for interchanging the subindexes 2 and 3.

Case 3. $T_2\times[\frac{1}{2},1]$ and $T_3\times[\frac{1}{2},1]$ are isotopic. Similar to Case 1, we can find a vertical convex annulus B connecting $T_2\times\{\frac{1}{2}\}$ and $T_3\times\{\frac{1}{2}\}$ with no ∂-parallel dividing curves that has Legendrian boundary intersecting the dividing sets of these tori efficiently. Extend B across $T_2\times[\frac{1}{4},\frac{1}{2}]$ and $T_3\times[\frac{1}{4},\frac{1}{2}]$ to a convex annulus $\widetilde{B}$ connecting $T_2\times\{\frac{1}{4}\}$ and $T_3\times\{\frac{1}{4}\}$ with Legendrian boundary intersecting the dividing sets of these two tori efficiently. For reasons similar to above, neither component of $\widetilde{B}\setminus B$ has ∂-parallel dividing curves. Thus, $\widetilde{B}$ has no ∂-parallel dividing curves. Cut $(\Sigma\times S^1)\setminus(T_1\times[0,1)\cup T_2\times[0,\frac{1}{4})\cup T_3\times[0,\frac{1}{4}))$ along $\widetilde{B}$, and round the edges. We get a thickened torus $T_1\times[1,2]$ embedded in $\Sigma\times S^1$ with convex boundary. The dividing set of $T_1\times\{2\}$ consists two circles of slope -1. Now we can see that the thickened torus $T_1\times[0,2]$ has I-twisting at least π since the dividing curves of $T_1\times\{\frac{1}{2}\}$ and $T_1\times\{2\}$ have slope -1 and those of $T_3\times\{1\}$ have slope ∞. Thus, $\Sigma\times S^1$ is inappropriate. This is again a contradiction.

Thus, $M=M(\frac{q_1}{p_1},\frac{q_2}{p_2},\frac{q_3}{p_3})$ admits no tight contact structures for which there exists a Legendrian vertical circle with twisting number 0. □

4. The $e_0=-1$ Case

Since part (1) of Theorem 1.5 is already proved, we will concentrate on parts (2) and (3) of Theorem 1.5. We will refine the method used in the $e_0\leq-2$ case to prove these results. Lemmata 4.1, 4.2 below and Lemma 3.5 from last section will be the main technical tools used in the proof.

Lemma 4.1. *Let ξ be an appropriate contact structure on $\Sigma\times S^1$. Suppose that $-\partial\Sigma\times S^1=T_1+T_2+T_3$ is convex and such that each of T_1 and T_2 has two dividing curves of slope -1, and T_3 has two horizontal dividing curves. Assume that there are pairwise disjoint collar neighborhoods $T_i\times I$ of T_i in $\Sigma\times S^1$ for $i=1,2,3$, such that $T_i\times\{0\}=T_i$ and $T_i\times\{1\}$ is convex with two vertical dividing curves. Then $(T_1\times I,\xi|_{T_1\times I})$, $(T_2\times I,\xi|_{T_2\times I})$ and $(T_3\times I,\xi|_{T_3\times I})$ are all basic slices, and the signs of these basic slices can not be all the same, where the sign convention of $(T_i\times I,\xi|_{T_i\times I})$ is given by choosing the vector associated with $T_i\times\{1\}$ to be $(0,1)^T$.*

Proof. Since ξ is appropriate, each $(T_i\times I,\xi|_{T_i\times I})$ is minimal twisting. From the boundary condition of these slices, we can see these are all basic slices. Assume that all these basic slices have the same sign. Then we have that $(T_1\times I,\xi|_{T_1\times I})$ and $(T_2\times I,\xi|_{T_2\times I})$ are isotopic relative to boundary. We isotope T_1 and T_2 slightly so that they have vertical Legendrian rulings. By Lemma 3.5, we can then find a properly

embedded convex vertical annulus A disjoint from $T_3 \times I$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap T_1) \cup (A \cap T_2)$ intersects the dividing sets of T_1 and T_2 efficiently. Cut $\Sigma \times S^1$ open along A, we get a thickened torus $T_3 \times [0, 2]$ such that each of $T_3 \times \{0\}$, $T_3 \times \{1\}$ and $T_3 \times \{2\}$ is convex with two dividing curves, and the slopes of the dividing curves are 0, ∞ and 1, respectively. Note that the slice $(T_3 \times [1, 2], \xi|_{T_3 \times [1,2]})$ has the sign different from that of $(T_1 \times I, \xi|_{T_1 \times I})$, and the slice $(T_3 \times [0, 1], \xi|_{T_3 \times [0,1]})$ has the same sign as that of $(T_1 \times I, \xi|_{T_1 \times I})$. So $\xi_{T_3 \times [0,2]}$ is overtwisted. This is a contradiction. Thus, the signs of the basic slices $(T_1 \times I, \xi|_{T_1 \times I})$, $(T_2 \times I, \xi|_{T_2 \times I})$ and $(T_3 \times I, \xi|_{T_3 \times I})$ can not be all the same. □

The following lemma is a special case of Lemma 37 of [7]. Its proof is quite similar to that of Lemma 3.5 ([7], Lemma 36). We will only give a sketch of it.

Lemma 4.2 ([7], Lemma 37)**.** *Let ξ be an appropriate contact structure on $\Sigma \times S^1$. Suppose that $-\partial\Sigma \times S^1 = T_1 + T_2 + T_3$ is convex and such that T_1 has vertical Legendrian rulings and two dividing curves of slope $-\frac{1}{n}$, where $n \in \mathbf{Z}^{>0}$, T_2 has vertical Legendrian rulings and two dividing curves of slope $\frac{1}{n}$, and T_3 has two vertical dividing curves. Let $T_1 \times I$ and $T_2 \times I$ be collar neighborhoods of T_1 and T_2 that are mutually disjoint and disjoint from T_3, and such that, for $i = 1, 2$, $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is convex with dividing set consisting of two vertical circles. If basic slices of $(T_1 \times I, \xi|_{T_1 \times I})$ and $(T_2 \times I, \xi|_{T_2 \times I})$ are all of the same sign, then there exists a properly embedded convex vertical annulus A with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap T_1) \cup (A \cap T_2)$ intersects the dividing sets of T_1 and T_2 efficiently.*

Sketch of proof. Similar to the proof of Lemma 3.5, we can show that, if we prescribe the sign of the basic slices of $(T_1 \times I, \xi|_{T_1 \times I})$ and $(T_2 \times I, \xi|_{T_2 \times I})$, then, up to isotopy that fixes T_1, T_2 and leaves T_3 invariant, there are at most two appropriate contact structures on $\Sigma \times S^1$ that satisfy the given conditions, each of which corresponds to one of the two diagrams in Figure 1. Since the two layers $T_1 \times I$ and $T_2 \times I$ are not contactomorphic, we can not find a contactomorphism between these two possible appropriate contact structures as before. Instead, we will construct an appropriate contact structure corresponding to each of these two diagrams, and show that each of these admits an annulus with the required properties.

Now consider the tight contact thickened torus $(T_2 \times I, \xi|_{T_2 \times I})$. Like in the proof of Lemma 3.5, we can construct an appropriate contact structure on $\Sigma \times S^1$ satisfying the conditions in the lemma that admits an annulus A with the required properties by "digging out" a vertical Legendrian ruling of a torus in an I-invariant neighborhood of $T_2 \times \{0\}$ parallel to the the boundary. Indeed, both of the possible appropriate contact structures can be constructed this way. To see that, we isotope $T_2 \times \{0\}$ and $T_2 \times \{1\}$ lightly to T_2' and T_3' with the same dividing curves and horizontal Legendrian rulings. Then connect a Legendrian ruling of T_2' and a Legendrian ruling of T_3' by a horizontal convex annulus B. The dividing curves of B is given in Figure 3. We can choose the vertical Legendrian ruling to be dug out to intersect one of the two dividing curves of B. These two choices of the vertical Legendrian ruling correspond to the two possible layout of the dividing curves on Σ_0' in Figure 1, and, hence, gives the two possible appropriate contact structures on $\Sigma \times S^1$ satisfying the given conditions. □

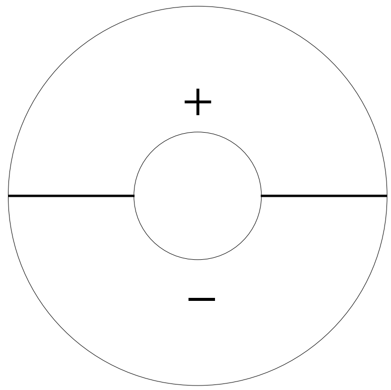

FIGURE 3.

Proof of (2) and (3) of Theorem 1.5. Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space such that $0 < q_1 < p_1$, $0 < q_2 < p_2$ and $-p_3 < q_3 < 0$. For $i = 1, 2, 3$, let $V_i = D^2 \times S^1$, and identify ∂V_i with $\mathbf{R^2}/\mathbf{Z^2}$ by identifying a meridian $\partial D^2 \times \{\text{pt}\}$ with $(1,0)^T$ and a longitude $\{\text{pt}\} \times S^1$ with $(0,1)^T$. Choose $u_i, v_i \in \mathbf{Z}$ such that $0 < u_i < p_i$ and $p_i v_i + q_i u_i = 1$ for $i = 1, 2, 3$. Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \to T_i$ by

$$\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & v_i \end{pmatrix},$$

where T_i and the coordinates on it are defined above. Then

$$M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).$$

Assume that ξ is a tight contact structure on M for which there exits a Legendrian vertical circle L in M with twisting number $t(L) = 0$. We isotope ξ to make $L = \{\text{pt}\} \times S^1 \subset \Sigma \times S^1$, and each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $n_i < 0$, i.e., ∂V_i is convex with two dividing curves each of which has slope $\frac{1}{n_i}$ when measured in the coordinates of ∂V_i given above. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \frac{-n_i q_i + v_i}{n_i p_i + u_i} = -\frac{q_i}{p_i} + \frac{1}{p_i(n_i p_i + u_i)}.$$

Then $-1 \le s_1, s_2 \le 0$ and $0 \le s_3 < 1$. As before, we can find pairwise disjoint collar neighborhoods $T_i \times I$'s in $\Sigma \times S^1$ of T_i's, such that $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is convex with dividing set consisting of two vertical circles.

We now prove part (2).

Assume that $q_3 = -1$, $\frac{q_1}{p_1} < \frac{1}{2p_3 - 1}$ and $\frac{q_2}{p_2} < \frac{1}{2p_3}$. By choosing $n_i \ll -1$, we can make $-\frac{1}{2p_3-1} < s_1 < -\frac{q_1}{p_1}$, $-\frac{1}{2p_3} < s_2 < -\frac{q_2}{p_2}$ and $\frac{1}{p_3+1} < s_3 < \frac{1}{p_3}$. So there is a convex torus in $T_i \times I$ parallel to the boundary, say $T_i' = T_i \times \{\frac{1}{2}\}$, that has two dividing curves of slope $-\frac{1}{2p_3-1}$, $-\frac{1}{2p_3}$ and $\frac{1}{p_3+1}$ for $i = 1, 2$ and 3, respectively. Let's consider the layers $T_i \times [\frac{1}{2}, 1]$. $T_1 \times [\frac{1}{2}, 1]$ is a continued fraction block consisting of $2p_3 - 1$ basic

slices. $T_2 \times [\frac{1}{2}, 1]$ is a continued fraction block consisting of $2p_3$ basic slices. $T_3 \times [\frac{1}{2}, 1]$ consists of 2 continued fraction blocks, each of which is a basic slice. We can find a convex torus $T_i'' = T_i \times \{\frac{3}{4}\}$ in $T_i \times [\frac{1}{2}, 1]$ parallel to bounary with two dividing curves of slope -1 for $i = 1, 2$, and 0 for $i = 3$.

Let the sign of the basic slice $T_3 \times [\frac{3}{4}, 1]$ be $\sigma \in \{+, -\}$. Note that, when $q_3 = -1$, then diffeomorphism $\varphi_3 : \partial V_3 \to T_3$ is given by

$$\varphi_3 = \begin{pmatrix} p_3 & p_3 - 1 \\ 1 & 1 \end{pmatrix}.$$

So the slopes 0 and $\frac{1}{p_3+1}$ of the dividing sets of T_3'' and T_3' correspond to twisting numbers -1 and -2 of Legendrian circles isotopic to the $-\frac{1}{p_3}$-singular fiber. And the basic slice $T_3 \times [\frac{1}{2}, \frac{3}{4}]$ corresponds to a stabilization of a Legendrian circle isotopic to the $-\frac{1}{p_3}$-singular fiber. Since we can freely choose the sign of such a stabilization, we can make the sign of the basic slice $T_3 \times [\frac{1}{2}, \frac{3}{4}]$ to be σ, too.

According to Lemma 4.1, the sign of the basic slices $T_i \times [\frac{3}{4}, 1]$ can not be all the same. Note that we can shuffle the signs of basic slices in a continued fraction block. So at least one of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ consists only of basic slices of sign $-\sigma$.

Case 1. Assume that all the basic slices in $T_1 \times [\frac{1}{2}, 1]$ are of the sign $-\sigma$. If $T_2 \times [\frac{1}{2}, 1]$ contains p_3 basic slices of the sign $-\sigma$, then we shuffle these signs to the p_3 slices closest to $T_2 \times \{1\}$. Now consider the thickened tori $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$ formed by the unions the p_3 basic slices closest to $T_1 \times \{1\}$ and $T_2 \times \{1\}$ in $T_1 \times I$ and $T_2 \times I$, respectively. Remove from M the open solid tori bounded by $T_1 \times \{\frac{5}{8}\}$, $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{1\}$. We apply Lemma 3.5 to the resulting $\Sigma \times S^1$ and the thickened tori $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap (T_1 \times \{\frac{5}{8}\})) \cup (A \cap (T_2 \times \{\frac{5}{8}\}))$ intersects the dividing sets of $T_1 \times \{\frac{5}{8}\}$ and $T_2 \times \{\frac{5}{8}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A and round the edges, we get a torus convex $\widetilde{T}_3$ isotopic to T_3 with two dividing curves of slope $\frac{1}{p_3}$. This means there exists a thickening $\widetilde{V}_3$ of V_3 with convex boundary $\partial \widetilde{V}_3$ that has two dividing curves isotopic to a meridian. Then $\xi|_{\partial \widetilde{V}_3}$ is overtwisted. This contradicts the tightness of ξ.

If $T_2 \times [\frac{1}{2}, 1]$ contains $p_3 + 1$ basic slices of the sign σ, then we shuffle all these signs to the $p_3 + 1$ slices closest to $T_2 \times \{1\}$. Let $T_2 \times [\frac{5}{8}, 1]$ be the union of these $p_3 + 1$ basic slices. Remove from M the open solid tori bounded by $T_1 \times \{1\}$, $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{\frac{1}{2}\}$. Apply Lemma 4.2 to the resulting $\Sigma \times S^1$ and the thickened tori $T_2 \times [\frac{5}{8}, 1]$ and $T_3 \times [\frac{1}{2}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap (T_2 \times \{\frac{5}{8}\})) \cup (A \cap (T_3 \times \{\frac{1}{2}\}))$ intersects the dividing sets of $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{\frac{1}{2}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A, we get a thickened torus $T_1 \times [1, 2]$ embedded in $\Sigma \times S^1$ that has convex boundary such that $T_1 \times \{1\}$ has vertical dividing curves and $T_1 \times \{2\}$ has dividing curves of slope $-\frac{1}{p_3+1}$. Then the thickened torus $T_1 \times [\frac{1}{2}, 2] = (T_1 \times [\frac{1}{2}, 1]) \cup (T_1 \times [1, 2])$ has I-twisting at least π. This again contradicts the tightness of ξ.

But $T_2 \times [\frac{1}{2}, 1]$ is a continued fraction block consisting of $2p_3$ basic slices. So it contains either p_3 basic slices of the sign $-\sigma$ or $p_3 + 1$ basic slices of the sign σ. So, the basic slices in $T_1 \times [\frac{1}{2}, 1]$ can not be all of the sign $-\sigma$.

Case 2. Assume that all the basic slices in $T_2 \times [\frac{1}{2}, 1]$ are of the sign $-\sigma$. If $T_1 \times [\frac{1}{2}, 1]$ contains either p_3 basic slices of the sign $-\sigma$ or $p_3 + 1$ basic slices of the sign σ, then there will be a contradiction just like in Case 1. So the only possible scenario is that $T_1 \times [\frac{1}{2}, 1]$ contains $p_3 - 1$ basic slices of the sign $-\sigma$ and p_3 basic slices of the sign σ. Now we shuffle all the $-\sigma$ signs in $T_1 \times [\frac{1}{2}, 1]$ to the $p_3 - 1$ basic slices closest to $T_1 \times \{1\}$. Now let $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$ be the unions the $p_3 - 1$ basic slices closest to $T_1 \times \{1\}$ and $T_2 \times \{1\}$ in $T_1 \times I$ and $T_2 \times I$. Remove from M the open solid tori bounded by $T_1 \times \{\frac{5}{8}\}$, $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{1\}$, and apply Lemma 3.5 to the resulting $\Sigma \times S^1$ and the thickened tori $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap (T_1 \times \{\frac{5}{8}\})) \cup (A \cap (T_2 \times \{\frac{5}{8}\}))$ intersects the dividing sets of $T_1 \times \{\frac{5}{8}\}$ and $T_2 \times \{\frac{5}{8}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A and round the edges, we get a torus convex $\widetilde{T}_3$ isotopic to T_3 with two dividing curves of slope $\frac{1}{p_3 - 1}$. This means we can thicken V_3 to a standard neighborhood $\widetilde{V}_3$ of a Legendrian circle isotopic to the $-\frac{1}{p_3}$-singular fiber with twisting number 0. Stabilize this Legendrian circle twice. We get a thickened torus $\widetilde{T}_3 \times [\frac{1}{2}, 2]$ such that $\widetilde{T}_3 \times \{2\} = \widetilde{T}_3$, $\widetilde{T}_3 \times \{\frac{3}{4}\}$ has two dividing curves of slope 0, and $\widetilde{T}_3 \times \{\frac{1}{2}\}$ has two dividing curves of slope $\frac{1}{p_3+1}$. Since we can choose the sign of these stabilizations freely, we can make both basic slices $\widetilde{T}_3 \times [\frac{1}{2}, \frac{3}{4}]$ and $\widetilde{T}_3 \times [\frac{3}{4}, 2]$ to have the sign $-\sigma$. There exists a convex torus, say $\widetilde{T}_3 \times \{1\}$, in $\widetilde{T}_3 \times [\frac{3}{4}, 2]$ parallel to boundary that has two vertical dividing curves. Use $\widetilde{T}_3 \times \{1\}$, we can thicken $T_1 \times [\frac{1}{2}, \frac{5}{8}]$ to $\widetilde{T}_1 \times [\frac{1}{2}, 1]$, such that $\widetilde{T}_1 \times [\frac{1}{2}, \frac{5}{8}] = T_1 \times [\frac{1}{2}, \frac{5}{8}]$, and $\widetilde{T}_1 \times \{1\}$ is convex with two vertical dividing curves. Since the basic $\widetilde{T}_3 \times [\frac{3}{4}, 2]$ has the sign $-\sigma$, all the basic slices in $\widetilde{T}_1 \times [\frac{5}{8}, 1]$ have the sign σ. Also note that all the basic slices in $\widetilde{T}_1 \times [\frac{1}{2}, \frac{5}{8}] = T_1 \times [\frac{1}{2}, \frac{5}{8}]$ have the sign σ. So we are now in a situation where the basic slices $\widetilde{T}_3 \times [\frac{1}{2}, \frac{3}{4}]$ and $\widetilde{T}_3 \times [\frac{3}{4}, 1]$ both have the sign $-\sigma$, and all the basic slices in $\widetilde{T}_1 \times [\frac{1}{2}, 1]$ have the sign σ. After we thicken $T_2 \times [\frac{1}{2}, \frac{5}{8}]$ to $\widetilde{T}_2 \times [\frac{1}{2}, 1]$, where $\widetilde{T}_2 \times \{1\}$ is convex with two vertical dividing curves, we are back to Case 1, which is shown to be impossible. Thus, the basic slices in $T_2 \times [\frac{1}{2}, 1]$ can not be all of the sign $-\sigma$ either.

But, as we mentioned above, one of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ have to consist only of basic slices of sign $-\sigma$. This is a contradiction. Thus, no such ξ exists on M, and, hence, we proved part (2) of Theorem 1.5.

It remains to prove part(3) now.

Assume that $q_1 = q_2 = 1$ and $p_1, p_2 > 2m$, where $m = -\lfloor \frac{p_3}{q_3} \rfloor$. By choosing $n_i \ll -1$, we can make $-\frac{1}{2m} < s_1 < -\frac{1}{p_1}$, $-\frac{1}{2m} < s_2 < -\frac{1}{p_2}$, and $0 < s_3 < -\frac{q_3}{p_3}$. Similar to the proof of part (2), we can find convex a torus $T_i' = T_i \times \{\frac{1}{2}\}$ in $T_i \times I$ parallel to boundary with two dividing curves that have slope $-\frac{1}{2m}$ for $i = 1, 2$, and 0 for $i = 3$. Then each of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ is a continued fraction block consists of $2m$

basic slices. And $T_3 \times [\frac{1}{2}, 1]$ is a basic slice. Let the sign of the basic slice $T_3 \times [\frac{1}{2}, 1]$ be $\sigma \in \{+, -\}$. For reasons similar to above, at least one of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ can not contain basic slices of the sign σ. Without loss of generality, we assume that all basic slices in $T_1 \times [\frac{1}{2}, 1]$ are of the sign $-\sigma$. We now consider the signs of the basic slices in $T_2 \times [\frac{1}{2}, 1]$.

Case 1. Assume that $T_2 \times [\frac{1}{2}, 1]$ contains m basic slices of the sign $-\sigma$. Then we shuffle these signs to the m basic slices in $T_2 \times [\frac{1}{2}, 1]$ closest to $T_2 \times \{1\}$. Denote by $T_1 \times [\frac{3}{4}, 1]$ and $T_2 \times [\frac{3}{4}, 1]$ the unions of the m basic slices in $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ closest to $T_1 \times \{1\}$ and $T_2 \times \{1\}$, respectively. Remove from M the open solid tori bounded by $T_1 \times \{\frac{3}{4}\}$, $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{1\}$, and apply Lemma 3.5 to the resulting $\Sigma \times S^1$ and the thickened tori $T_1 \times [\frac{3}{4}, 1]$ and $T_2 \times [\frac{3}{4}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap (T_1 \times \{\frac{3}{4}\})) \cup (A \cap (T_2 \times \{\frac{3}{4}\}))$ intersects the dividing sets of $T_1 \times \{\frac{3}{4}\}$ and $T_2 \times \{\frac{3}{4}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A and round the edges, we get a thickened torus $T_3 \times [1, 2]$ with convex bounary such that $T_3 \times \{1\}$ has two dividing curves of slope ∞, and $T_3 \times \{2\}$ has two dividing curves of slope $\frac{1}{m}$. Note that $\frac{1}{m} \leq -\frac{q_3}{p_3}$. If $\frac{1}{m} = -\frac{q_3}{p_3}$, then, as above, the existence of $T_3 \times [1, 2]$ means that we can thicken V_3 to $\widetilde{V}_3$ such that $\xi|_{\widetilde{V}_3}$ is overtwisted, which contradicts the tightness of ξ. If $\frac{1}{m} < -\frac{q_3}{p_3}$, we can choose s_3 so that $\frac{1}{m} < s_3 < -\frac{q_3}{p_3}$. Then the thickened torus $T_3 \times [0, 2] = (T_3 \times I) \cup (T_3 \times [1, 2])$ has I-twisting greater than π, which again contradicts the tightness of ξ. So $T_2 \times [\frac{1}{2}, 1]$ can not contain m basic slices of the sign $-\sigma$.

Case 2. Assume that $T_2 \times [\frac{1}{2}, 1]$ contains $m+1$ basic slices of the sign σ. We shuffle one of the σ sign to the basic slice in $T_2 \times [\frac{1}{2}, 1]$ closest to $T_2 \times \{1\}$,and denote by $T_2 \times [\frac{3}{4}, 1]$ this basic slice. Similar to the proof of Theorem 1.4, we can find a convex vertical annulus A in M satisfying:

(1) A has no ∂-parallel dividing curves;
(2) $\partial A = (A \cap (T_2 \times \{\frac{3}{4}\})) \cup (A \cap (T_3 \times \{\frac{1}{2}\}))$, which is Legendrian and intersects the dividing sets of $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{\frac{1}{2}\}$ efficiently;
(3) A is disjoint from T_1 and the interior of the solid tori in M bounded by $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{\frac{1}{2}\}$.

Note that, since $q_1 = 1$, the diffeomorphism $\varphi_1 : \partial V_1 \to T_1$ is given by

$$\varphi_1 = \begin{pmatrix} p_1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Remove from M the interior of the solid tori in M bounded by $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{\frac{1}{2}\}$, and cut it open along A. We get a thickening $\widetilde{V}_1$ of V_1, whose boundary is convex with two dividing curves of slope ∞. Then $\widetilde{V}_1$ is a standard neighborhood of a Legendrian circle isotopic to the $\frac{1}{p_1}$-singular fiber with twisting number 0. We stabilize this Legendrian circle once. This gives a thickened torus $\widetilde{T}_1 \times [0, 2]$ with convex boundary such that $\widetilde{T}_1 \times \{2\} = \varphi_1(\partial \widetilde{V}_1)$, which has two dividing curves of slope 0, and $\widetilde{T}_1 \times \{0\}$ has two dividing curves of slope $-\frac{1}{p_1 - 1}$. Since we can choose the sign

of the stabilization, we can make the sign of this basic slice σ. Since $-\frac{1}{p_1-1} \geq -\frac{1}{2m}$, we can find tori $\widetilde{T}_1 \times \{\frac{1}{2}\}$ and $\widetilde{T}_1 \times \{1\}$ in $\widetilde{T}_1 \times [0,2]$ parallel to the boundary such that $\widetilde{T}_1 \times \{\frac{1}{2}\}$ has two dividing curves of slope $-\frac{1}{2m}$, and $\widetilde{T}_1 \times \{1\}$ has two dividing curves of slope ∞. Note that $\widetilde{T}_1 \times [\frac{1}{2}, 1]$ is now a continued fraction block consisting of $2m$ basic slices of the sign σ. Now use $\widetilde{T}_1 \times \{1\}$ to thicken $T_2 \times [\frac{1}{2}, \frac{3}{4}]$ to $\widetilde{T}_2 \times [\frac{1}{2}, 1]$ such that $\widetilde{T}_2 \times [\frac{1}{2}, \frac{3}{4}] = T_2 \times [\frac{1}{2}, \frac{3}{4}]$, and $\widetilde{T}_2 \times \{1\}$ has two vertical dividing curves. Note that $\widetilde{T}_2 \times [\frac{1}{2}, 1]$ is a continued fraction block that contains at least m basic slices of the sign σ. Now, similar to Case 1, we can find a contradiction. Thus, $T_2 \times [\frac{1}{2}, 1]$ can not contain $m+1$ basic slices of the sign σ either.

But $T_2 \times [\frac{1}{2}, 1]$ contains $2m$ basic slices. So either m of these are of the sign $-\sigma$, or $m+1$ of these are of the sign σ. This is a contradiction. Thus, no such ξ exists on M, and, hence, we proved part (3) of Theorem 1.5. □

5. Final Remarks

When a Seifert space M has more than 3 singular fibers, or the base surface has genus ≥ 1, there exists a vertical incompressible torus T embedded in M. Using this vertical incompressible torus T, we can construct infinitely many pairwise non-isomorphic universally tight contact structure on M. (See [14] for more details.) For infinitely many universally tight contact structures constructed this way, there is a tubular neighborhood $T \times I$ of T in M that has I-twisting equal to π. In such a neighborhood, we can always find a convex torus T' isotopic to T whose dividing set consists of vertical circles. By the Legendrian Realization Principle, we can realize a vertical circle on T' disjoint from the dividing set as a Legendrian curve. This Legendrian curve is a vertical Legendrian circle with twisting number 0. So any such "larger" Seifert space admit tight contact structures for which there exists a Legendrian vertical circle with twisting number 0.

Using Theorem 1.4 and Ghiggini and Schönenberger's result that, when $r \leq \frac{1}{5}$, no tight contact structures on the small Seifert space $M(r, \frac{1}{3}, -\frac{1}{2})$ admit Legendrian vertical circles with twisting number 0, one can easily prove that the Brieskorn homology spheres $\pm\Sigma(2,3,p)$ do not admit tight contact structures for which there exists Legendrian vertical circles with twisting number 0. It's certainly very interesting to see if this is true for all small Brieskorn homology spheres. Unfortunately, Theorem 1.5 is not strong enough to apply to this case. We will have to develop new techniques before we can tackles this problem.

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